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ZPC and ZPD: Zones of Teaching and Learning

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The goal of this article is to examine students' mathematical development that occurs as a teacher works within each of 2 zones of learning: students' zones of proximal development (ZPD) and students' zones of potential construction (ZPC). ZPD, proposed by Vygotsky, is grounded in a social constructivist perspective on learning, whereas ZPC, proposed by Steffe, is grounded in a radical constructivist perspective on learning. In this article, we consider potential pragmatic differences between ZPD and ZPC as instantiated during a semester-long teaching experiment with 2 Grade 6 students. In particular, we examine the constructions that a teacher fostered by working with these students in each zone of learning. The data suggest that operating in their ZPD but outside of their ZPC impacts the learning opportunities and resulting constructions of the students. Finally, we characterize aspects of ZPD and teacher assistance that foster the development of mathematical concepts.

Key words: Constructivism; Fractions; Learning; Learning theories; Piaget; Teaching effectiveness; Vygotsky

Constructivism has been used as a rationale for using cooperative groups, technology, whole-class discussions, and various other teaching and learning techniques. The implication in many cases is that these techniques facilitate students' constructions, which might otherwise be replaced by passive absorption. However, radical constructivists and social constructivists alike understand that constructing knowledge is not a choice for students or their teachers to make. In all of their actions and experiences, students are constructing, even when they are taking notes or reiterating a teacher's remarks (Noddings, 1990). The question is a matter of *what* they are constructing (Boaler, 2000).

From two different perspectives on learning, we have Vygotsky's (1986) theoretical framework describing the social construction of knowledge and von Glasersfeld's (1995) radical constructivist theory describing the construction of cognitive schemes and operations. The differences in epistemology between the two theories have spurred arguments over whether the mind exists in the individual or in society (Cobb, 1994; Lerman, 1996; Steffe & Thompson, 2002). But researchers from both perspectives are concerned with development that occurs as a result of an individual's interactions with the environment and others. It goes beyond the scope of this article to describe these perspectives in great detail. Instead we refer the reader to the original works of these researchers (Piaget, 1950; von Glasersfeld,

1995; Vygotsky 1986; also, Confrey, 1994, 1995 provide elegant summaries) and highlight only the main ideas underlying two related constructs: the zone of proximal development (ZPD) proposed by Vygotsky (1978) and the zone of potential construction (ZPC) proposed by Steffe (1991).

In this article we consider potential pragmatic differences between ZPD and ZPC as instantiated during a semester-long teaching experiment with two Grade 6 students. In particular, we examine the constructions that a teacher fostered by working with these students in each zone of learning. We then characterize aspects of ZPD and teacher assistance that foster the development of mathematical concepts.

ZONE OF PROXIMAL DEVELOPMENT (ZPD)

> The zone of proximal development of a child is the distance between her actual development level as determined by independent problem solving and her level of potential development as determined through problem solving under the guidance or in collaboration with more capable peers. (Vygotsky, 1978, p. 86)

Vygotsky defined a child's ZPD in terms of success in solving a set of problems predetermined by the teacher. Students can solve some problems independently; other problems are so unintelligible to the student that she cannot even solve them with guidance. The ZPD consists of those problems that a student may solve when "provided with some slight assistance, [such as] the first step in a solution or a leading question" (1986, p. 187). The teacher's job in facilitating learning is to pose problems in that zone and provide assistance. The underlying assumption is that students learn through assistance, resulting in subsequent development that will enable them to independently interpret and solve such problems later: "Therefore the only good kind of instruction is that which marches ahead of development and leads it" (p. 188).

The kind of teacher assistance that Vygotsky (1986) advocated even extends to imitation, but not the kind of simple imitation that one might use in training an animal. "Intelligent, conscious imitation comes instantly in the form of insight, not requiring repetition" (p. 188). It is a kind of imitation that, according to Vygotsky, is strictly human and stands opposed to "persistent training," which results in meaningless mechanical actions or "habits" (p. 188). "To imitate, it is necessary to possess the means of stepping from something one knows to something new" (p. 187). In other words, a student must have already developed certain functions to meaningfully interpret and imitate a new function.

This claim raises the question of how a teacher can know whether a student has meaningfully imitated an action or whether she has simply mechanically repeated observed action that she was trained to follow. When does a teacher's assistance generate meaningless habits, and when does it promote development? Vygotsky (1978, 1986) would make the distinction based on whether the imitated action can be generalized or transferred to other, similar situations. How general must students' actions be in order for the teacher to infer meaningful learning has occurred? And what kind of assistance should the teacher provide to ensure that learning will lead to mathematical development? To begin answering these questions, we examine

the various forms of assistance present in the research of various neo-Vygotskians. But, first, we contrast the foregoing description of ZPD with a description of ZPC.

ZONE OF POTENTIAL CONSTRUCTION (ZPC)

Schemes describe students' cognitive structures that develop through the abstraction of actions and operations; they are teacher constructs that provide "a way to discuss the development of stable and predictable courses of action" (Confrey, 1994, p. 4). A student's ZPC is also a teacher construct, embedded in the scheme theoretic perspective of learning. Steffe defined ZPC as the range "determined by the modifications of a concept a student might make in, or as a result of, interactive communication in a mathematical environment" (1991, p. 193). This distinction points to a fundamental epistemological difference between the social constructivist and the radical constructivist perspectives: In Vygotsky's view, problems, concepts, and other forms of knowledge exist in society first before being internalized by the individual; from a scheme theoretic perspective, problems, concepts, knowledge, and even society itself have their beginnings in the unique experiences and constructions of the individual (von Glasersfeld, 1995). The former perspective beholds "[students] growing into the intellectual life of those around them" (Vygotsky, 1978, p. 88). The latter perspective obliges the teacher to consider differences between students' conceptions, or schemes.

Teachers can only know their students through the models they build of them. ZPC is modeled by the teacher in hypothesizing what schemes and operations might become available to a student through a reorganization of schemes and operations within the teacher's existing model of the student (Steffe & D'Ambrosio, 1995). When a teacher infers that two students are thinking in compatible ways about a mathematical problem, it only means that the teacher's models of the two students' actions cannot be distinguished or that the teacher has decided that the differences are unimportant in a particular context. Whereas this admits a limitation of teachers' possible understanding of students, it may be a useful limitation because it allows the teacher to act effectively within a classroom of 30 minds as if there were only a few minds, at least in consideration of a particular mathematical concept.

Because ZPCs are constructed by teachers in terms of potential reorganizations to models of students' cognitive structures and operations, task design must also depend on particular models of students. In utilizing ZPD, the target tasks are taken as given, and the question for the teacher is one of how much support students need to solve the tasks. So, then, how might instruction differ when posed tasks fit one zone versus the other zone?

FORMS OF ASSISTANCE IN ZPD

Several neo-Vygotskian researchers (Bruner, 1985; Cole, 1985; Goos, Galbraith, & Renshaw; 2002; Tharp & Gallimore, 1988; Wertsch & Stone, 1985; Wertsch & Toma, 1995) have worked to elaborate on ZPD and describe the teacher's role in

facilitating development within it. These researchers' perspectives provide a wide range of interpretations of ZPD and teacher assistance, which sometimes allude to formalist, innativist, and even behaviorist assumptions about mathematical learning. For instance, Cole (1985) described learning as "the acquisition of culturally appropriate behavior" (p. 158), and Tharp and Gallimore (1988) validated the use of standardized assessments, like the Stanford-Binet, in measuring development as "conventional and correct" (p. 30). At best, these tests inform teachers whether their students can engage with particular formal ideas, such as the orientation of directions on a compass; these tests provide nothing from which to infer students' unique mathematical constructions.

In *Rousing Minds to Life*, Tharp and Gallimore (1988) described two stages of ZPD leading to the internalization of performance. The first stage requires assistance from others; the second stage moves toward internalization through self-assistance, such as the use of self-directed speech. The descriptions are wrought with innativist terms, as in the idea that learning *awakens* the development of mental functions (p. 31). "Awakening" suggests that all mental functions exist within the brain from birth and are developed through physical maturation that is stimulated by experience. Indeed, the authors cite Vygotsky in asserting that "[ZPD] defines those functions that have not yet matured but are in the process of maturation, functions that will mature tomorrow but are currently in the embryonic state" (p. 30). In contrast, Wertsch and Stone (1985) emphasized internalization as the process by which functions are transformed from the social plane to *create* a plane of individual consciousness.

Scaffolding plays a key role in Tharp and Gallimore's (1988) first stage of ZPD, "where performance is assisted by more capable others" (p. 33). Whereas "behavior shaping" follows the behaviorist model of "simplifying a task by breaking it down into a series of steps," scaffolding is intended to simplify only the child's role in solving the original task (p. 33). Beyond selecting an appropriate task to begin with, a teacher can provide such assistance by selecting appropriate tools, directing the student's attention, "holding important information in memory," and offering encouragement (p. 34). But the authors' description of scaffolding seems to fall back into a behaviorist paradigm when they describe "structuring tasks into sub-goals and sub-sub-goals . . . until the entire script is assembled back from its parts" (p. 34).

Although Tharp and Gallimore emphasized the importance of teachers "achieving intersubjectivity" with their students, this discussion is limited to sharing and adjusting goals, and the authors refer to students' knowledge as "subordinate to the structures of the academic discipline that is being transmitted" (p. 35). In other words, the authors, once again, indicate a formalist perspective of disciplines like mathematics, while opening the door for a behaviorist interpretation of ZPD and scaffolding. For example, Carnine (1997) cited Tharp and Gallimore (1988) to support his interpretation of scaffolding as "telling students what to do, step by step" (p. 135). He illustrated a mathematical example of this in which a teacher instructed a student solving a ratio problem. The teacher asked the student to "find the question you're going to answer," "write the units and the numbers for the ratio," and

"find the other number that is given and use it in the other ratio," until the initial problem was reduced to a familiar exercise (p. 134).

Bruner (1985) brought clarity to the idea of assisting a learner in the learner's ZPD by explaining that the teacher "serves the learner as a vicarious form of consciousness until he is able to master his own action through his own consciousness and control" (p. 24). Thus, Bruner understood the goal of learning as the development of consciousness in a particular domain. He recognized that the domain of primary concern for Vygotsky was language acquisition, and he argued that language acquisition "is at least partly innate" (p. 28). Bruner also understood mathematics as a language, but he was primarily concerned with students' problem-solving activity.

Bruner used an example of problem solving in which young children were learning to build a pyramid from blocks. Assistance in completing the task included "constructing the pyramid slowly, with conspicuous marking of the subassemblies," thus "segmenting the task," "minimizing the possibility of error," and "reducing the number of degrees of freedom the child must manage" (p. 29). These descriptions of assistance might be interpreted as an endorsement of direct instruction, but Bruner was careful to note that "instruction in words comes only after the child knows how to do the problem" (p. 30). He envisioned teachers setting goals for students, enticing them into activities related to those goals, allowing students to do what they can in those activities, and "filling in the rest" (p. 29).

Bruner described the teacher as the only one with conscious control over the activity and the goal (p. 31). On the other hand, Goos et al. (2002) studied collaborative ZPDs in which groups of students negotiated goals without the aid of an expert. Collaborative ZPDs were created by the groups as the students challenged each other with their ideas, "making students' thinking public and open to critical scrutiny" (p. 219). Perhaps because the groups diminished the role of the teacher as a shaper of behaviors and designer of goals, Goos et al. recognized that scaffolding can be detrimental to students' development because "it may deny students the opportunity to resolve their own difficulties" (p. 220).

Other characterizations of ZPD challenge the merits of scaffolding because it seems to propagate a particular view of knowledge and learning. For example, Newman and Holzman (1997) rejected the idea of knowledge altogether, viewing learning as societal developments of creative performance rather than the accumulation or construction of knowledge: ZPD "offers us much more than a better understanding of epistemological dualism; it provides us with a general framework for going beyond modern knowing altogether to a postmodern activity-theoretic form of understanding" (p. 45). We do not address such revolutionary perspectives in this article but note that the motivation for the authors' critique of the duality of knowledge seems similar to that of radical constructivism.

METHODOLOGY

The data used for the analysis presented in this article were generated for a larger study (Norton, 2004). Excerpts from that study are used for this particular analysis

of students' ZPDs and ZPCs. The methodology of the larger study is described here in order to give the reader a sense of the intense experience of the teacher with the children in planning, implementing, and analyzing instructional episodes.

The semester-long study consisted of a teaching experiment with three pairs of Grade 6 students; pairing the students made it possible to conduct analyses of individual cognition while affording the opportunity to study student-student interactions. The goal was to build second-order models of students' fractional reasoning and other constructions relevant to the larger study (specifically, students' conjecturing activity). Because of the extensive data involving student-student and student-teacher interactions, analysis of ZPDs and ZPCs became feasible.

Teaching experiment methodology requires a particular approach to teaching in which the teacher must "continually establish meaning of the students' language and actions" (Steffe & Thompson, 2000, p. 11) so that the students' actions guide the teacher in his attempts to "become the students and to think as they do" (p. 13). This approach is important on two levels. First, by continually establishing meaning, the teacher is developing new hypotheses about students' cognition while remaining open to surprises. Second, attempting to think as students do, the teacher is in a position to understand the students' ways of operating and compare them to his or her own in order to design tasks to provoke creative activity in the students. On both levels, the teacher experiences constraints in building viable models and meaningful tasks based on the dichotomy of expected and observed activities of students. This feedback provides the guiding principle for hypothesis testing and the design of new tasks within and between protocols.

Data collection occurred between February and May 2003. The first author, as teacher, conducted 20-minute initial interviews with 12 children from three different Grade 6 classrooms. He then chose 6 students to form three pairs, based on his initial assessment of their ways of operating with fractions. Two of the pairs were formed based on the students' part-whole operations with fractions; a third pair was formed based on those students' iterative operations with fractions. For the analysis presented in this article, we describe only the interactions of the third pair of students, the pair that best exemplified potential differences between ZPD and ZPC.

During the episodes of their teaching experiment, the pair of students (pseudonyms Will and Hillary) sat to one side of the researcher in front of the computer. Each student had a mouse to use; the two mouses had been spliced so that either student could control the cursor on the screen. Two cameras were behind the students, one focused on the two students and the teacher, the other zoomed in on the computer screen. An observer sat behind the group taking notes and monitoring the two cameras. The observer was a graduate student in mathematics education who was interested in the study and was able to commit time participating in and discussing the teaching experiment with the researcher. His job was to observe student-teacher interactions and provide feedback to the teacher on their effectiveness, especially concerning task design.

The tasks were designed using TIMA: Bars (Olive & Steffe, 1994), computer software that consists of many microworlds (the plurality refers to the potential for

students' creation of problematic situations within the program). This program provided the medium for students' activity and for posing tasks in this research project. The microworlds allows students to enact operations on rectangular bars of varying sizes and shapes, which are created by clicking and dragging the computer mouse. "Making a bar along with possible actions . . . can [also] be used to engender certain conceptual operations" (Steffe & Olive, 1996, p. 177). The *possible actions* within the microworlds include MAKE, COPY, PULLOUT, ROTATE, JOIN, REPEAT, PARTS, SHADE, BREAK, WIPE, UNIT [whole] BAR, and MEASURE. We describe these actions during our analysis of the teaching experiment.

The first episode or two with Will and Hillary was dedicated to free play, allowing the students to exercise their ways of operating using the actions available in the microworlds. Goal-directed activity followed from play through tasks that either the students or the teacher posed, based on experiences in play and students' available operations. In between episodes, the teacher analyzed students' available schemes, their conjectures, and the effectiveness of the posed tasks. Based on this analysis, he planned follow-up activities for the following episode.

Initial analysis consisted of two major components that lasted the duration of the school's semester: building second-order models of students' operations and examining the role of the teacher interacting with the students. This analysis occurred in between episodes (including debriefing with the observer) and was the foundation for the design of tasks during the teaching experiment. Once the teaching experiment was complete, there were about 50 episodes recorded on about 100 mini DV tapes that needed to be coded. The goals in this second phase of analysis were to describe the kinds of activity during each segment of the episodes and to establish models of operations and schemes accounting for students' actions. The data for Will and Hillary were reanalyzed for this article with a lens towards interpreting the interactions themselves and the occasions when students were operating in their ZPDs or their ZPCs.

The teacher was primarily concerned with students' development of schemes and operations, so he focused on working within the students' ZPCs. However, many of the student-student and student-teacher interactions can be interpreted as occurring within the students' ZPDs. We will examine the various interpretations of ZPD that would support this claim and, in conclusion, use these interpretations to characterize interpretations of ZPD that are useful in supporting students' development of schemes and operations. Many of the interactions can be interpreted within Will's ZPD but not within his ZPC, although they were within Hillary's ZPD and ZPC. We focus on these interactions, but first we consider an initial model of the students' ways of operating, which is necessary to make inferences about students' ZPCs.

INITIAL MODELS OF THE STUDENTS

The initial interviews and first few teaching episodes provided the teacher with an opportunity to develop initial models of the students' schemes and operations, while providing the students with opportunities to become familiar with the

microworlds. During the initial interview, Will demonstrated that he could interpret fractions as part-whole relations. For example, he was able to produce 2/3 of a given bar by partitioning it into three equal parts and pulling out two of them. The mental operations associated with such actions are those of a *part-whole fractional scheme*. Beyond that, given a unit fractional part (such as 1/4 of a bar) and an unpartitioned whole bar, Will could determine the size of the part relative to the whole by iterating the part (four times) to reproduce the whole. This indicates that Will had also constructed a *partitive unit fractional scheme* with which to conceptualize unit fractional parts through a multiplicative relationship with the whole. However, Will could not generalize this way of conceptualizing unit fractional parts to nonunit fractional parts. That is to say, he lacked a (more general) *partitive fractional scheme*. For example, in attempting to produce 2/3 from a 6/6 bar, Will pulled three parts from the six and then pulled two of those, thinking, "Maybe it will come out to be 2/3." If he had developed a partitive fractional scheme, we would expect him to anticipate that 2/3 would be greater than half the size of the 6/6 bar, and thus more than half of its parts. Instead, this example illustrates a novel use of Will's part-whole fractional scheme.

Will also struggled to explain the equivalence of fractions like 2/3 and 6/9. In his attempts to do so, he drew a 2/3 bar and a 6/9 bar, shading in two out of three equal parts and six out of nine equal parts, respectively. But the 6/9 bar was drawn much bigger, which he justified by saying, "Because there are nine parts in that one and only three in the other one." Will's operations with fractions did not include the constraint of preserving the whole (also indicated by his previous attempt to produce 2/3 from 6/6). On the other hand, Hillary had developed a partitive fractional scheme, and a *commensurate fractional scheme* (with which to produce and explain fractions with equivalent measures) was in her ZPC. We make this claim because, in our model of Hillary up to this point in the teaching experiment, we saw potential for a hypothetical reorganization (accommodation) of her partitive fractional scheme, engendered by problem-solving activity, and resulting in the new way of operating.

A *reversible partitive fractional scheme* was also in Hillary's ZPC and ZPD after the first few episodes. In other words, the tasks posed by the teacher engendered modifications of her partitive fractional scheme; she was beginning to use her partitive fractional scheme in reverse to reproduce wholes from fractional parts. There was indication of this even during her initial interview when the teacher asked her to produce a bar such that the given bar would be five times as big. She immediately produced 1/5 of the given bar indicating a *splitting operation*, which requires a student to use partitioning to resolve a situation that is iterative in nature (partitioning a whole into five parts to find one that iterates five times to reproduce the whole). Although she could split, Hillary struggled in the next task in which the teacher asked her to reproduce the whole using a 3/4 part, with no visible partitions and no visible whole. She confused the given part and desired whole, producing 3/4 of the given part.

There is one more operation described in the analysis section, an operation that both students had developed before the teaching experiment began: *units coordi-*

nation at three levels. This operation describes the students' ability to produce a unit of units of units. For instance, Hillary and Will could consider 12 as a unit of 3 units of 4. However, only Hillary seemed to be able to produce such units with fractions, such as considering the whole as three units of 1/3, each consisting of four 12ths. Will's apparent inability to produce such units seems to be related to his difficulty in working with nonunit and commensurate fractions.

ANALYSIS OF TEACHING EPISODES

Protocol 1: Hillary's Production of 2/3 From 12 Parts

We begin our analysis in the middle of the fourth teaching episode. By this point, both students appeared comfortable working with the microworlds. Hillary had been trying to make a bar that would measure "2/3" relative to a given whole bar, without partitioning the whole bar into three parts. Will suggested that she try using 12 parts, presumably because he recognized that 2 and 3 each divide 12. Hillary's actions in attempting to use 12 parts to produce 2/3 are recorded in Protocol 1, and her resulting productions are illustrated in Figure 1.

H: [Pulls 4/12 and places it in the bottom third of the whole bar, then the middle third, then the top third]

W: [Following Hillary's actions] One . . . two . . . three!

H: That's two thirds! [Referring to the 4/12]

T: You think you've got two thirds?

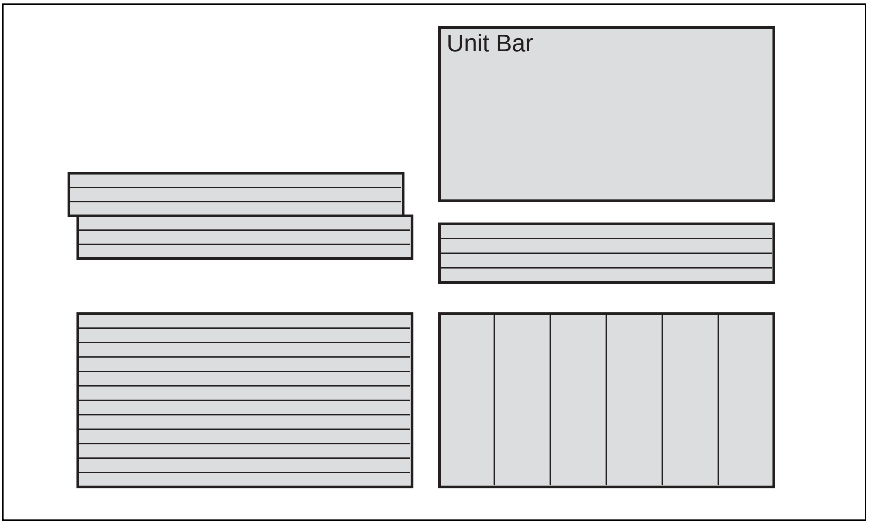

Figure 1. Hillary's production of 4/12.

H: [Nods]

T: Okay. What makes you think you've got two thirds?

H: You can put. . . . Oh, never mind, that's not two thirds.

T: What is it?

W: I think it would be *one* third.

H and W: [Simultaneously] It's one third.

T: Oh! It's *one* third. If you've got *one* third, can you think of a way to get *two* thirds?

H: Mm-hmm. [Begins counting eight parts in the 12/12 bar]

As she began to justify her production of 2/3, Hillary realized that she had in fact produced 1/3; she knew that 1/3 was the fraction you could put into the whole bar three times. She also knew that 2/3 was 1/3 iterated twice. After Hillary placed the 4/12 bar in the whole bar three times, Will also realized that Hillary had made 1/3. He seemed to have a concept of 1/3 that was similar to Hillary's, but he did not seem to understand 2/3 as she did. These facts highlight Will's limitations with nonunit fractional parts: They were not partitive fractions but part-whole fractions. As the protocol continued, Will reverted to his part-whole fractional scheme as he had in past protocols involving nonunit fractions.

W: I was going to say she could take two out of that [pointing to a 3/12 bar that Hillary had pulled (but not named) by mistake] and put it into the whole bar and see if it would make two thirds.

T: Oh. What would happen though?

W: It would be more than *one* third.

The stark contrast between Will's reasoning with unit fractions and nonunit fractions was apparent as he suggested that he might create 2/3 by pulling two of the three 12ths. He was able to treat the two parts as a unit and compare it to the whole. He was subsequently able to iterate the two parts in the whole, treating them as a unit fraction and realizing that they would go into the whole more than three times. But Will's confusion continued as Hillary attempted to justify her production of a nonunit fraction, 2/3, from 8/12, as illustrated in Figure 2.

H: [Finishes pulling out 8/12 and starts to move it to the bottom 2/3 of the whole bar] *That's* two thirds. I had four and . . .

W: Nope.

H: . . . four was one third of that. So you add double the number, so I got eight, so that's two thirds of [the whole bar].

T: [To Will] What do you think about her explanation?

W: I don't think so because if you put [8/12] again into [the whole bar], it would be over it. [Begins counting eight parts from the bottom of the 12/12 bar and whispers] She took out eight.

H: If you put [4/12] right there [in the top third of the whole bar] in that one third.

T: [To Will] Look what she just did.

W: She just made another whole. Yep. I think it might be two thirds.

T: So what was confusing you at first, Will?

W: What was confusing me at first was that if she put [8/12] in there again, it would go over.

T: So if you put two thirds in there twice, it's okay if it goes over?
H: Yeah. The other half wouldn't matter.
T: [To Will] So what should be left over if you put two thirds in [the whole bar]?
W: Just four little parts like [the 4/12].
T: Which is how much out of the whole?
W: Uh, one third.

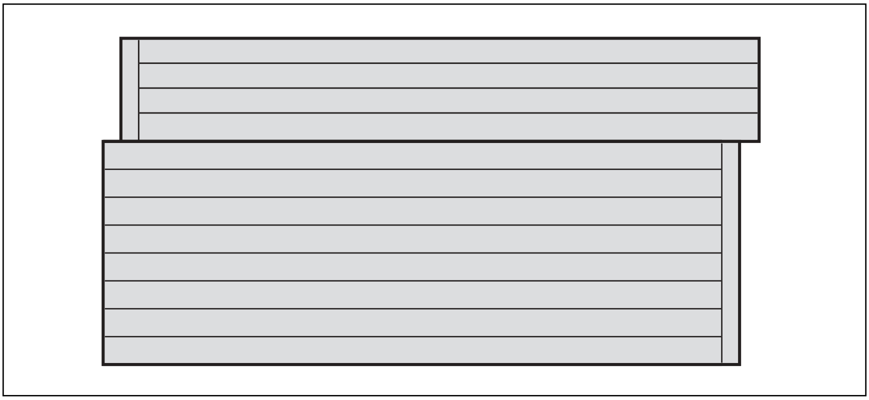

Figure 2. Hillary's reconstruction of the whole bar.

From her explanation ("Four was 1/3 of that. So you add double the number, so I got eight, so that's 2/3."), it is clear that Hillary knew that 2/3 was two of 1/3. Later in the segment, she even demonstrated her understanding that 1/3 was half of 2/3 when she said, "The other half wouldn't matter," indicating a reversible partitive fractional scheme. Her explanation also affirms that she used units coordination at three levels, uniting 4/12 as 1/3 and iterating it.

Hillary's action of putting the last 4/12 bar into the whole bar to complete it was accepted by Will and convinced him that Hillary's production of 8/12 was indeed 2/3. This is an example in which Will was able to assimilate Hillary's actions and make local adjustments to his fractional schemes and concepts. It may be that Will could understand 2/3 as the complement of 1/3 (as Hillary could), but his hesitation in naming the 4/12 bar as 1/3 (at the end of the segment) is a counterindication of that. If Will's actions are consistent with his way of operating all along in the episodes to this point, he is most likely seeing 12 parts as composed of 8 parts and 4 parts. Will seemed to have accepted Hillary's solution to the situation with Hillary's guidance, but it has yet to be seen whether Will developed new ways of operating with fractions.

Whereas this episode seemed to be situated in Hillary's ZPC, in which her ways of operating were challenged and resolved, Will assimilated the situation and

Hillary's actions using his partitive unit fractional scheme (interpreting 2/3 as something that should iterate into the whole three times) and his part-whole fractional scheme (eventually making sense of Hillary's production by considering the four additional parts that would complete the whole). His ways of operating were not challenged in a way that engendered an accommodation of those schemes; we were not working in his ZPC. The possibility that we were working in his ZPD, however, is considered in the concluding section.

Will seemed convinced by Hillary and able to describe her actions, suggesting an understanding of this particular problem, but it is not clear whether he supported his solution with fraction operations. Given his struggle to consider the numbers of parts in determining the size of the whole and his willingness to create new wholes in order to show nonunit fractions, it seems more likely that Will's solutions were procedures that were constructed based on his whole-number knowledge and interpretations of his social interactions with Hillary. In fact, in the next exchange, Will focuses on the whole numbers in the numerators and denominators to find a pattern that would explain the equivalent fractions.

T: Why would 8 out of 12 give us two thirds?

W: Well, 3 goes into 12, no 2 goes . . . no . . . I was going to say something goes into 12 three times. That's the reason why I told her to go to 12, because the last number right there [pointing to the "3" in the "2/3" displayed in the measure box] could end up a 3 because something like that goes into 12. . . . Eight twelfths is the same as two thirds.

T: How do you know?

H: [Hillary, who had been in silent reflection for about a minute (but still appeared to follow my discussion with Will), turns from looking at the screen and smiles.] Um. . . .

T: [Writes 8/12 = 2/3 on the chalkboard and repeats the question for Will, ignoring Hillary for the moment.]

W: Four times 2 is 8 and then 4 times 3 is 12.

When the teacher asked Will to explain why 8/12 would measure as 2/3, he used a procedure for determining the equivalence of fractional numerals, involving whole-number operations of multiplication and division: a *procedural scheme for producing equivalent fractions*. The scheme did not seem to account for commensurability (in terms of fractional sizes) and had not yet been used to *produce* new fractions other than those commensurate with 1/2. The scheme involved generating post hoc explanations for the common measures of nonunit fractions. Apparently, Will had not used uniting or iteration operations for nonunit fractions. The fact that he did not immediately know what "something" went into 12 three times would be unlikely if he had constructed 2/3 as a unit of two 1/3s, each composed of 4/12.

Will's partitive unit fractional scheme established fractions as the reciprocal of the number of iterations needed to reproduce the whole bar. Because 2/3 was a nonunit fraction, he did not know whether it would be established by two or three iterations. With this in mind, Will suggested that Hillary use 12/12 in their attempts to produce 2/3. Whereas his explanation in the latest segment focuses on finding a number divisible by 3, in previous segments he had expressed interest in finding a fraction that would go into the whole twice. In the end, he found a number divis-

ible by 2 and 3. Will had reasoned similarly in choosing 6/6 in a previous episode but had begun to consider the role of denominator ("last number") of nonunit fractions more.

Hillary, on the other hand, seemed to have been constructing new ways of operating with fractions. Toward the end of Protocol 1, she had begun to excitedly express an idea, but the teacher was focused on Will's explanation at the time. It is unclear whether her initial idea was related to the 6/6 bar, but the observer interjected with a question about it, and Hillary appeared equally excited to consider that question.

Protocol 2: Hillary's Explicit Use of Commensurate Fraction Operations

O: Is it even possible to do it using sixths [referring to the 6/6 bar still displayed on the screen]?

H: Yeah there is!

W: Let's see. There's two thirds right here [pointing to the 8/12 bar] and you took out eight. You could add some. [After several seconds of looking at the screen. . . .] *Is* there a way to do it?

H: Uh-huh. What I've been thinking about is six can. . . . There's two parts, every two parts is one third [pointing her thumb and index finger to the three pairs of sixths in the 6/6 bar] and if you put two parts together [pointing across four sixths] that's going to make two thirds [pointing to the whole bar].

W: That's what I was thinking about.

T: Okay. You do it, Will.

W: [Pulls four sixths from the 6/6 bar and measures it to reveal "2/3."]

Hillary seemed to have constructed the operations of a commensurate fractional scheme. Her expression that "every two parts is 1/3," without pulling any of the sixths, demonstrates that she could perceive 6/6 as a unit of units of sixths that could be partitioned to form a unit of three units, each containing two sixths. Hillary may have recognized the novelty of this way of operating, which would explain her sporadically heightened level of excitement throughout the last several minutes of the episode.

In suggesting that Hillary "add some" to the 6/6 bar, Will seemed to have invented a rule that eight parts, regardless of their size relative to the 12/12 bar, would constitute 2/3. However, he was able to make meaning of Hillary's expressions and complete her suggested production. Again, this suggests that the teacher (and Hillary, as a more capable peer) were working within Will's ZPD, but he did not seem to be operating in his ZPC. (We examine this possibility as Case B in the concluding section.) Although Hillary could coordinate three levels of fractional units to create new fractions, Will could only act on experiential units of units of units once they were established. In this case, Hillary created two out of the three units of sixths, and, once she verbalized this, Will could use the parts on the screen to follow her instruction.

The episode concluded as Will and Hillary tried to produce 2/3 with different numbers of parts. Will suggested that they use 16ths, and, "in a way," Hillary thought this might work. Will was still not focused on size; he initially thought that 8/16

might be 2/3 before realizing that it would be 1/2. He may have been using the number 8 in attempting to produce 2/3, simply because eight parts had produced 2/3 in the case of 12ths. Similar to his initial approach in trying to produce 2/3 using 12ths, he then tried half of the eight 16ths. Although he had been able to interpret Hillary's previous comments regarding the production of 4/6 as 2/3 (and carry out her intended actions!), Will could not act creatively with a commensurate fractional scheme of his own and instead resorted to using his whole-number procedures again. This suggests that although he may have learned something in working with Hillary within his ZPD, he had not yet developed an independent way of operating as she could.

Protocol 3: Hillary's Novel Use of Uniting Operations

In the next episode, Hillary and Will struggled to make sense of improper fractions. From the students' classroom experience, they were proficient in applying an algorithm to convert improper fractional numerals to mixed numbers. After Will had successfully produced 10/3, Hillary calculated that it should be 3 and 1/3, and the teacher asked her to show that. Hillary lined up a 3/3 bar with a 1/3 bar (as displayed in the lower-left corner of Figure 3) and claimed that was 3 and 1/3!

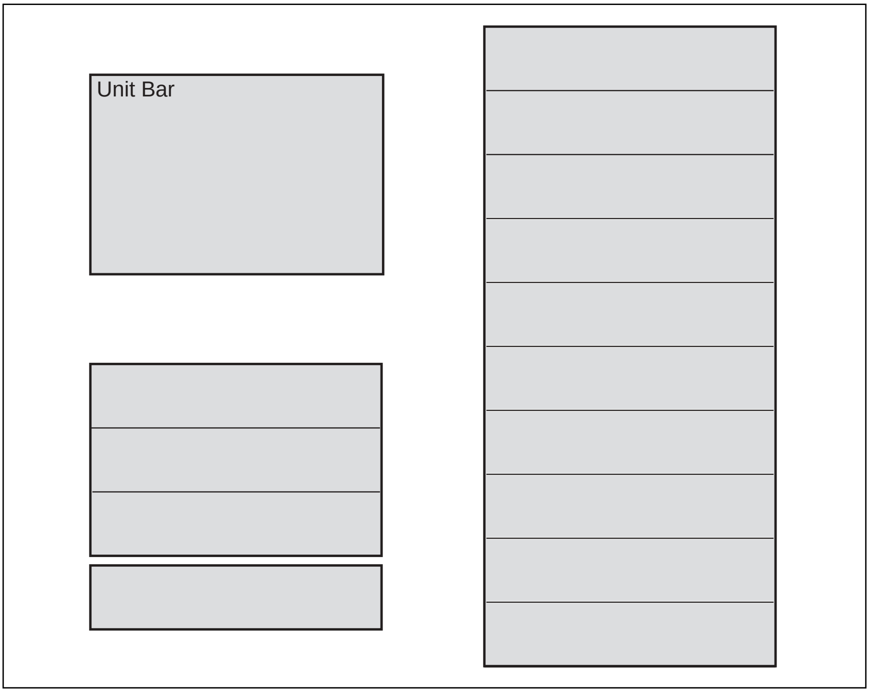

Figure 3. Hillary's production of 3 and 1/3.

H: Three and one third?
T: Where's three and one third?
H: This is three [sweeping the cursor over the 3/3 bar] and that's one third [pointing with the cursor to the 1/3 bar].
T: Is it the same as ten thirds [pointing at the 10/3 bar and then looking at Hillary]?
H: [Looks at the screen]
W: [Shakes his head *no*] Mm-mm.
T: Doesn't look the same. . . . This one [pointing to the 10/3 bar] looks a lot bigger!
W: I think it's three and one third, but it's not the same as three tenths or ten thirds or whatever.
H: We're going to take three of these bars [circling the cursor around 3/3 at a time, down the 10/3 bar].
T: Do you want to fill them in to show?
H: [Fills in three of the 1/3 bars at a time, with each group of three a different color] Three and three. . .
W: [Pointing at the groups] That's three, three, and three and we've got one left over.
T: Is that three and one third? You said three, three times.
W: It could be nine and one third.
H: [After a few seconds pause looking at the screen] Three whole bars and one third out of it.
T: Did you hear what she said? Does that make sense?
W: Yeah.
T: So, explain in your own words, Will, what she means in terms of this.
W: Those three, right there, are whole bars. There's three of them; so there's three whole bars. See [dragging the whole bar over each]. They're the same size as the whole bar.
T: So this right here is not three and one third [pointing to the lower-left corner of the screen displayed in Figure 3], is it?
W: [Pauses and shakes his head "no"] Uh-uh.
T: What is it?
H: [Pauses for a couple of seconds] One and one third?
W: [Joins the 3/3 and 1/3 bars and measures "4/3."] Four-thirds is the same as one and one third.

Hillary and Will used their conversion algorithm treating the fractional units as units of one until the teacher questioned them about the sizes of the bars: "Is it the same as 10/3?" Even then, only Hillary deduced that the bars should be the same size. Will seemed to see no problem in the different sizes associated with the "10/3" and "3 and 1/3" fractions, even though he had established that one could be converted into the other. He could make meaning of Hillary's subsequent actions and explanations, but his meaning did not seem to be based on uniting fractional wholes within the bar, and he was not constructing the parts as fractions either. Rather, he first claimed that the "three, three, and three" might make "9 and 1/3," and later used the whole bar to show that the whole fit into 10/3 three times. He had not recognized the copies of the whole bar within 10/3 until Hillary had colored them.

Will's ways of reasoning throughout this protocol continues to be consistent with his work up to this point in which he focused on counting the number of parts. He

has still not pulled parts together to form a unit and hence has not been successful in iterating them in order to form a nonunit fraction or a whole. Nor has he been successful in iterating his collection of parts, which is not yet a unit, to form what Hillary understands as three wholes, as needed in this protocol. We might say, then, that Hillary and Will were solving two very different problems: Hillary's problem involved fractions, whereas Will's problem involved only whole numbers.

Hillary eventually united the three 3/3 parts as a composite whole using SHADE and dragging the whole bar to help her enact the novelty. This was a novel use of her uniting operation and units coordination, as indicated by her tentative assertion about her initial production: "One and 1/3?" It seems that she was actualizing a new way of operating that was in her ZPC.

The episode continued as Will asked Hillary to produce 7/6. Initially, she partitioned the whole bar into seven parts, but quickly changed this to sixths, with no intervention from Will or the teacher. She made a copy of the whole bar, partitioned it into six parts, pulled one more, and joined to make 7/6. Will agreed that Hillary had produced 7/6 and immediately knew that it would convert to 1 and 1/6. He produced the mixed number by making a new copy of the whole bar, partitioning it into six parts (saying, "that's one"), and pulling another sixth: "That's going to be 1 and 1/6." After Hillary agreed that he was right, Will measured each of the two bars in turn to reveal "1" and "1/6."

Will's production constituted a new way of acting for him, since he seemed confident in renaming the 6/6 as a whole. Yet, it is unclear whether he has created a unit of sixths or is still thinking of the six parts as wholes making a new unit of wholes, which he calls "one." The question remains whether he had assembled a novel way of operating based on his interpretation of Hillary's activity. In that case (Case C), Will had been able to explain Hillary's actions in his own words. Following Hillary's production of 7/6, he had also explicitly established that "one" referred to one whole bar and, of course, he knew how to produce 1/6. He had also conceptualized "1 and 1/6" as "1" and "1/6." What remains uncertain is whether he could unite composite wholes within a given improper fraction, with the goal of reconstituting the fraction as a mixed number. In other words, could he operate independently in a manner compatible with the way Hillary could?

Protocol 4: Will's Ambiguous Sense of Improper Fractions

Will's actions later in the episode indicate that he could not conceive of 7/6 as seven iterations of 1/6 of the whole, as Hillary could. Hillary posed 16/5 to Will. In response, he made a copy of the whole bar, partitioned it into 16 parts, pulled 5 parts, and joined them to make 21 16ths.

W: [Finishing his production] That's going to be sixteen fifths.

T: All right, let's check. What do you think Hillary?

H: It's not.

W: [Measures to reveal "21/16"] Twenty-one sixteenths?! Okay. . . .

T: [Laughing] Why did it come out to be twenty-one sixteenths?

H: It's because he used 16 parts instead of 5.

T: Do you get what she's saying?

W: Yeah. That's where I messed up; I cut the unit wrong.

T: Yeah. Ya'll both switch it sometimes, so you have to be careful. So, what should you cut it into?

W: Fifths.

T: Why fifths?

W: 'Cause it's sixteen fifths.

Will's actions indicate that he still lacked a fraction scheme for producing improper fractions. He seemed to be using procedural schemes, invented in the social context of assimilating Hillary's actions, to relate the numerators and denominators in such fractions. In past protocols, he used procedures successfully, but it was evident that his actions did not represent schemes for meaningful fractional operation. Although he could act (with assistance) as Hillary had and even explain her productions, he had not developed ways of operating that were compatible with hers.

In this case, Will's assimilation of Hillary's actions in producing 7/6 resulted in a procedure that simply concatenated the number of parts indicated by the numerator and denominator. He had partitioned a copy of the whole bar just as Hillary had, except that he used 16 parts, whereas Hillary would have used 5; he then pulled a number of parts to join as Hillary had except that Will used the number in the denominator instead of building up to the number in the numerator. So, in assimilating Hillary's action, he did not distinguish the unique roles of numerator and denominator as one would in using a partitive fractional scheme. Hillary knew that Will's production was not appropriate and reminded him that "16/5" referred to the numerosity of fifths making up the fraction. Will understood this "'cause it's 16/5."

Once the students had successfully produced 16/5, the teacher asked Hillary to find the mixed number. The whole bar was not partitioned, and Hillary dragged the whole bar into the 16/5 bar three times (top, middle, and bottom) before the teacher suggested that she use FILL to keep track, as she had done with 10/3. When she was done filling the three composite wholes, she claimed, "That's 3 and 1/5." After a moment of reflection, Will seemed to understand, saying, "There's three out of all of those, there's one left over, and there's five in each." Although Will interpreted Hillary's language and action appropriately, transforming 16/5 to 3 and 1/5, we cannot say that her way of operating was in his ZPC. This is because ZPC is not determined by social context but by a hypothetical reorganization of a students' present ways of operating. Hillary took the 5/5 bar as a given, which, coupled with her transformation of 16/5 to 3 and 1/5, indicates that she was working at the level of a unit of units of units. There is no indication that Will could operate independently with such units. The question remains as to whether we should consider this way of operating (Case D) to be in Will's ZPD.

Hillary seemed to have created new ways of operating through Protocol 1, and she used those new operations in Protocol 4. Using an unpartitioned whole bar, Hillary was able to determine the number of whole units in the improper fraction. We contend that the task offered Hillary opportunities to make new constructions,

so it was apparently situated in her ZPC. Qualitative differences between the students' ways of operating became more and more apparent throughout the teaching experiment. It also became more apparent that Will was assimilating Hillary's actions with his whole-number schemes rather than developing new ways of operating with fractions. This was most explicit in Protocol 5.

Protocol 5: Will's New Procedure for Reproducing Wholes From Nonunit Fractions

Will produced 9/13, covered the whole, and challenged Hillary to reproduce the whole from that fractional bar. Hillary then measured the given bar as "9/13." After about 10 seconds, she partitioned it into 9 parts, pulled one 13th, repeated it four times, and joined the parts to make the whole. When Hillary had completed her production, she checked the number of parts, determining that there were 13. This determination seemed to satisfy Hillary's goal, as she indicated that she was certain with no need for further checking. Next, Hillary posed 3/16 to Will, who had to reproduce the whole. Will measured the visible three-part bar (the original whole bar was covered) as "3/16," pulled one part, joined it to the others, and repeated the resulting four-part bar four times. Finally, he checked the number of parts in the production to find "16."

T: You got it?

W: [Nods affirmatively]

T: Let's measure and check . . .

W: [Measures "1"]

T: . . . and then you can uncover. There's lots of ways to check. You are getting more sure every time aren't you?

W: [Nods affirmatively, uncovers the original whole bar, and drags it over his production, apparently to compare sizes.]

T: All right. Good job!

O: How did you know how to do it that way?

W: 'Cause I got it off of Hillary's idea.

O: [Laughs] What was Hillary's idea?

W: To, uh. . . . Well, when we measured it, I did . . . I put down nine thirteenths. So, she measured it, and it was nine thirteenths. So, she just added four more [pointing to the last four parts in his production of 16/16].

When Hillary solved the problem that Will had posed, her goal seemed to be to make 13 parts because she knew that 9/13 was 9 of 1/13 and that 13 of these 1/13 units would recreate the whole. This sort of reasoning would involve a reversible partitive fractional scheme and units coordination of the fractions at three levels. In observing Hillary and subsequently solving a similar problem, Will had constructed a procedure by analogy. In other words, Will had used his whole-number operations in assimilating general steps to Hillary's solution: Hillary had *nine* 13ths and added 4 more to make *thirteen* 13ths; Will had been given *three* 16ths and knew he had to add on until he reached *sixteen* 16ths. Will had been very explicit about his analogy in explaining "Hillary's idea." It is also interesting to note that, although Will did not seem to coordinate fractions on three levels of units, he did use three

levels of units coordination in producing sixteen from four (producing a unit of 16 by iterating a unit of 4 parts four times).

We hypothesize that Will's actions were based on a procedural scheme that could emulate the operations of a reversible partitive fractional scheme: his *procedural scheme for reversing ratios*. The goal of this scheme was to produce the number of parts in the denominator from a given number of parts in the fraction bar. We will see further indication of Will's procedural scheme and affirmation of our hypothesis in the subsequent teaching episode. The constitutive characteristic of such a procedural scheme is that it is constructed in the context of assimilating the language and action of another student, using operations of a scheme different than the one used by the operating student. In this case, Will used the operation of his adding scheme for whole numbers in interpreting Hillary's language and action that were produced using her reversible partitive fractional scheme. Will could generalize some of the contextual details, such as the specific numbers in the fractional measure, but we will see that his procedure depended on other contextual details, such as starting from a *partitioned* fraction bar.

Protocol 6: A Test of Will's Procedural Scheme for Reversing Ratios

On April 14, we had our first teaching episode in 2 weeks, following the students' weeklong spring break. Although the students' actions in the episode often do not represent new ways of operating, we mention them to highlight the relative permanence of the student's schemes and operations. At the beginning of the episode, the teacher asked the students to remake a hidden whole bar given an unpartitioned bar that was 2/3 of it. Hillary immediately grabbed the mouse and dialed PARTS to "3." She hesitated a moment before dialing PARTS to "2," but then immediately partitioned the given bar into two parts, pulled one of them, and joined it to the others to make the whole. Despite the prolonged break from working with fraction bars or even with classroom mathematics, Hillary's reversible partitive fractional scheme was still available for solving such tasks.

Next, Hillary made a wiped (unpartitioned) 3/15 bar for Will, and he was supposed to reproduce the covered whole. After measuring the given bar to be "1/5," he partitioned it into two parts, pulled one of those parts, repeated it until he had three additional parts (for a total of five parts), and joined them.

W: I think that's it.

T: Okay. How did you think about that?

W: Well, I measured and it said one fifth, and so I pulled out one and I added [holding out three fingers on his left hand] about. . . . I think it was three more? Three or four more, and that equaled five, and that equals one whole.

T: Okay. Let me ask this. The first thing you did was you put the piece into two parts. Why did you do that?

W: Just so I could add. . . . So I won't have to. . . . So when I cut them down lower, I won't have to add as many on.

T: Okay. . . . Because you thought you would run out of room or something?

W: No. Because, see, if you cut it down like more [pointing to the bar he had just made]

> then it's going to be smaller than that, and since she covered up a whole lot [pointing to the cover over the whole], I figured she cut it down as bigger pieces.

Will had modified his procedural scheme for reversing ratios to account for his observations and interpretations of Hillary's actions, forming a new procedure. When Hillary was given a wiped 2/3 bar, she partitioned it into two parts and joined three copies to make the whole. Will's actions with 1/5 were analogous. Will's previously constructed numerical procedure, which had proven adequate for resolving such situations before spring break, had been unnecessarily abandoned, bringing into question whether such numerical procedures (procedural schemes) were permanent enough to call schemes. Moreover, it seems that Will's propensity for inventing new procedures was impeding his construction of a partitive fractional scheme. Will's modification did nothing to increase his power in operating with fractions. He conjectured that he should partition the given fraction into two parts before producing a total of five parts, supplanting his old procedure. The old procedure was not operationally flexible enough to survive.

At the end of the episode, the difference between Hillary and Will's ways of operating was yet again apparent. The teacher posed a wiped 5/6 bar to Will and asked him to remake the whole with Hillary's help. Will measured the bar and paused for several seconds, appearing to be stuck, so Hillary suggested that he partition the given bar into five parts. Will followed her suggestion and proceeded to add one more sixth to the 5/6 bar, but then continued repeating that sixth until he had produced 12/6. Hillary looked perplexed by this, and Will explained, "I was going to add five more to that 5 [pointing to the numerator of 5/6 in the measure box] and make it 10, and then six more to that 6 and make it 12, so that way it would be ten twelfths."

Will seemed to assume that he needed to create more parts to establish the equivalent fraction that he assumed the teacher had used to make the wiped 5/6 bar. This may explain why he did not use his procedural scheme for reversing ratios, which he had used effectively at least three times in the past. Hillary, on the other hand, had constructed a reversible partitive fractional scheme that was as effective in the new situations as it had been in situations from the previous episode. So, at least in that regard, Hillary's operative schemes were more permanent than Will's procedural schemes. Hillary seems to have created new ways of operating through Protocol 3 and used those new operations in Protocol 6. Using an unpartitioned whole bar, she was able to determine the number of units in the improper fraction.

Protocol 7: Will's Construction of 2/5 as a Partitive Fraction

Through April 14, Hillary and Will had participated in nine teaching episodes together, but Will had yet to develop ways of operating that were compatible with Hillary's. The teacher became aware that his interactions with Will (and those between Hillary and Will) were not engendering a partitive fractional scheme. Will's only way of working with nonunit fractions had been to treat them as part-whole ratios, often disregarding the role of the whole bar. So, the teacher determined that

Will should be challenged to compare the *sizes* of nonunit fractions in order to activate his partitive unit fractional scheme and generalize it to nonunit fractions, thus reorganizing the scheme as a partitive fractional scheme. For example, the teacher asked Will to make a fraction that was just a little bigger than 1/2. Will responded by partitioning the whole into two parts (halves), partitioning one of those parts into two parts (fourths), and joining one of them (1/4) to one of the first parts (1/2). However, he named the resulting (3/4) bar as "one half . . . because [he] didn't cut any more out than just half."

The teacher found opportunities to provoke conflict between products of Will's partitive unit fractional scheme and his part-whole fractional scheme—conflict that engendered a reorganization of those schemes, which resulted in Will's eventual development of a partitive fractional scheme. In the example described above, the teacher elicited Will's part-whole fractional scheme by having him compare the number of equal parts in the 3/4 bar and the whole bar, so that Will recognized that the bar was indeed 3/4. The first solid indication that the teacher's new efforts had been successful in engendering Will's development of a partitive fractional scheme occurred during the second to last teaching episode (on May 7, four episodes after April 15).

The teacher asked Will to produce a 1/5 bar from a wiped 2/5 bar. Will had already measured the bar as "2/5."

T: Can you use that [the wiped 2/5] to make one fifth?

W: [Dials PARTS to 2, partitions the given bar, pulls one out, releases the mouse and turns to look at me]

T: You sure?

W: [Nods]

T: How do you know?

W: Well, it's two fifths and if you cut it in half, it will take one off.

T: Oh, okay. Good.

Protocol 7 continued after Will had followed my instructions to reset the screen to the way it had been before the protocol began. I then asked him to make the whole from the wiped 2/5 bar. Will repeated the bar once, partitioned the result into four parts, pulled out two of the parts and lined them up with the others. When the observer asked him how much he had, Will thought for a moment and replied, "6/5." He completed his production by removing one of the sixths.

Something had changed in Will's ways of operating. Whereas in previous episodes a fraction like 2/5 only meant two of something (apparently unrelated to the size of the fraction bar), now Will was able to identify that 2/5 was two of 1/5, and he could identify the 1/5 within 2/5 by removing the other half of 2/5. Moreover, he did not lose track of the fact that he was operating with fifths even when he went on to produce 6/5 and the whole from 2/5! Tasks at the end of the teaching experiment took into account models of Will's ways of operating, and seemed to involve new ways of operating that *were* within his ZPC, engendering a partitive fractional scheme (i.e., provided opportunities for Will to construct the whole and nonunit fractions from iterations of a unit fraction).

DISCUSSION AND CONCLUSIONS

Hillary had constructed a partitive fractional scheme before the first teaching episode began. The teacher had also determined that a commensurate fractional scheme and a reversible partitive fractional scheme were in her ZPC. In fact, she actualized the new schemes through accommodations engendered by the problem-solving activity illustrated in Protocols 1 and 2, in which her ability to coordinate fractional units at three levels played a critical role. The relative permanence of her reversible partitive fractional scheme was indicated by her actions during Protocol 6 (immediately following spring break), and she had even begun constructing improper fractions and meaningfully converting them to mixed numbers during Protocol 3. Previous research has indicated that splitting operations and units coordination at three levels are critical in these constructions (Hackenberg, 2007; Olive & Steffe, 2002). We hypothesize that the availability of these operations differentiated the ZPCs of Hillary and Will.

The tasks that the teacher posed, along with other forms of assistance, were successful in provoking reorganizations in Hillary's ways of operating. So, we can say that the tasks were in her ZPD, although ZPDs do not necessarily involve a consideration of individualized tasks or mental operations, as do ZPCs. We will return to this distinction later, but we first consider the manner in which the teacher worked within Hillary's ZPD.

From a Vygotskian perspective, we can interpret the teacher's actions in Protocol 1 as assistance that contributed to Hillary's learning and her eventual development of a commensurate fractional scheme. First of all, the teacher posed a task (producing 2/3 without using three parts) that Hillary could meaningfully assimilate using her partitive fractional scheme, while challenging her to consider other fractions of equivalent sizes relative to the whole. Second, after Hillary produced a 4/12 bar, the teacher asked her to explain her reasoning ("What makes you think you have 2/3?"). This question seemed to focus Hillary's attention on the size of the bar that she had produced, at which point she and Will recognized that it was 1/3. Finally, the teacher asked whether Hillary could use *one* third to produce *two* thirds.

Hillary completed the task by producing an 8/12 stick. This indicates that she had learned how to produce commensurate fractions, with assistance from the teacher in the form of questioning. Although Will was able to respond to most of the questions as Hillary had, he did not have a partitive fractional scheme, nor unit coordination of fractions at three levels, which Hillary seemed to use in producing 8/12. By Protocol 2, we see that Hillary had developed an independent way of operating with commensurate fractions. Even in Protocol 1, she seemed to be explaining her reasoning quite independently, while providing assistance to Will. With the assistance of a more capable peer, Will seemed to learn how to act as Hillary did, but without a partitive fractional scheme or units coordination available, he did not develop a way of operating that was compatible with hers.

In Protocol 3, we see another example of Hillary's development through teacher-assisted learning. The teacher began by asking Hillary to show why 10/3 and "3

and 1/3" were the same. Once again, this task challenged Hillary to use her units coordination in a novel way. And once again, the teacher posed a series of follow up questions ("Where's 3 and 1/3"; "Is it the same as 1/3?") and suggestions ("Do you want to fill them in to show?") that supported her learning and her subsequent development of a new way of operating. In the next section, we consider why similar assistance did not promote or provoke Will's development of compatible ways of operating.

The Cases

Throughout Protocol 1, Will was able to assimilate Hillary's actions without conflict, until she produced a nonunit fraction (2/3). In fact, he was the first to correctly name her production of 1/3 from 4/12. This seemed to instantiate a collaborative ZPD (Goos et al., 2002) in which each student contributed toward the eventual production of 2/3 (8/12). However, Will did not have a partititve fractional scheme available with which to assimilate this production in a manner compatible with Hillary's partitive fractions. His independent strategy for producing 2/3 involved pulling two parts (12ths) from three parts—a clear indication that he was relying on a part-whole scheme to define nonunit fractions as ratios. Although Hillary subsequently convinced Will that the 8/12 bar was indeed 2/3, it was an explanation (adding on 4 more 12ths to produce the whole) that, once again, appealed to his part-whole scheme.

So, we have a case (Case A) in which the students seemed to work in a collaborative ZPD, followed by Hillary's assistance as a more capable peer (Tharp & Gallimore, 1988; Vygotsky, 1978, 1986), but Will did not develop independent ways of operating that were compatible with Hillary's. We might say that Hillary and Will had learned a concept on a social plane (Wertsch & Stone, 1985, p. 164), but it is clear that Will's internalization of it was very different from that of Hillary. He constructed a procedural scheme for producing equivalent fractions, whereas Hillary had constructed a commensurate fractional scheme. The former was based on whole-number knowledge; the latter was a generalization of Hillary's partitive fractional scheme.

The difference between the students' constructions was evident in Protocol 2, in which Will suggested that Hillary could "add some" to produce 2/3 from 6/6. Will's suggestion indicates that his procedural scheme did not generalize well to different situations. On the other hand, Hillary confidently and excitedly stated that every 2/6 was a 1/3 part, "And if you put two parts together, that's going to make two thirds." Once again, Will was able to meaningfully assimilate Hillary's explanation as he used it to produce the 4/6 bar that Hillary described (Case B). He had moved from "spectator" to "participant" (Tharp & Gallimore, 1988), but had yet to develop anything like a partitive fractional scheme, let alone a commensurate fractional scheme.

The difference between the students' constructions cannot be accounted for by considering their interactions alone. Researchers must build models of students'

current constructions in order to explain why one student seems to internalize actions differently from another (Steffe, 1999). We have explained these cases by noting that Hillary had constructed a partitive fractional scheme and units coordination at three levels, neither of which were available to Will for producing nonunit fractions through iteration. Will could use his partitive unit fractional scheme, part-whole scheme, and procedural schemes to assimilate Hillary's actions, but his productions did not represent partitive fractions, and he could not independently produce nonunit fractions. These conclusions are supported by Will's responses to tasks in Protocol 6, intended to test the teacher's model of Will's constructions. His responses indicated that he had constructed only procedural schemes.

In Protocol 3, Will could meaningfully interpret Hillary's explanation that 10/3 and 3 and 1/3 were equivalent by explaining in his own words. But we have noted (Case C) that Will and Hillary were solving two different problems: Hillary's was a fractions problem; Will's was a whole-number problem. If the adult or more expert peer is "the only one who knows the goal of the activity the two of them are engaged in" (Bruner, 1985, p. 31), Will and Hillary were, nonetheless, working in Will's ZPD. We found (in Protocol 4) that this engagement did not lead to the development of a general way of operating with improper fractions.

The teacher seemed to be working in Will's ZPD during Protocol 4, questioning him until he was able to explain that he should have used fifths, "because it's sixteen fifths." Many of the interactions between the teacher and Will, as well as Hillary and Will, used modeling, awards, and questioning to assist Will—methods that neo-Vygotskians describe as scaffolding techniques (Tharp & Gallimore, 1988). Moreover, Will was able to meaningfully interpret the actions of the teacher and Hillary. So, we can argue that Case D, and the interactions in all of the other cases, occurred within Will's ZPD. However, we found that many of these interactions engendered only procedural schemes, which were not as permanent or flexible as Hillary's operational schemes. We also found that Will did not develop a partitive fractional scheme until the teacher designed tasks that took into account Will's current ways of operating—not just his actions (independent or otherwise). In other words, the teacher needed to make inferences from Will's actions and posit a consistent and coherent model of his individual cognition. Although student-student and student-teacher interactions were critical to the students' learning in every case, these interactions could not, alone, explain the students' development.

Implications for Teaching and Research

We have examined several problematic situations that Will was able to resolve with some degree of assistance, either from the teacher or his more capable (or collaborative) peer. The forms of assistance included modeling, questioning, and praising, but they constituted mild assistance compared to the descriptions and suggestions of many neo-Vygotskians (Bruner, 1985; Cole, 1985; Tharp & Gallimore, 1988). And, until the end of the teaching experiment, they failed to promote the kinds of development that would have made Will's ways of operating

compatible with Hillary's. This is because the assistance provided by Hillary and the teacher had not taken Will's current ways of operating into account, only his perceived success in solving problems (independently or with assistance). Assistance was only successful when the teacher realized that Will's procedural schemes did not indicate operational development, at which point the teacher provided assistance within his ZPD *and* ZPC.

Neo-Vygotskians, as well as Vygotsky himself, have used the acquisition of language as the paradigm case for describing ZPDs (Bruner, 1985, p. 25). To the degree that mathematics is a language, their theory about mathematical learning is appropriate. But mathematics is also a science of numbers, patterns, and relationships (NCTM, 2000), and it involves a logical structure of internalized objects and operations (Piaget, 1970). As Hillary and Will have taught us, these objects and operations are very different than those that we take as shared on the social plane. Whereas Vygotsky recognized this in admitting that external and internal activity are related but not identical (Wertsch & Stone, 1985, pp. 163–7), some neo-Vygotskians have gone so far as to describe working within a student's ZPD as providing "consciousness for two" and "minimizing the cost or possibility of error" (Bruner, 1985, p. 29), with the goal of modifying student behaviors so that culturally determined correct behaviors are internalized (Cole, 1985). In examining ZPDs as apprenticeships, Cole (1985) explained that "virtually never is a novice permitted to engage in a task where costly failure is likely" (p. 158). Such descriptions applied to mathematical learning imply the following: that learning is not necessarily a goal-directed activity because the teacher can supply the goal (Bruner, 1985, p. 31), that the development of consciousness is the purpose of mathematical development as it is in language (Wertsch & Stone, 1985, p. 163), and that teachers should make mathematics less problematic for their students (Carnine, 1997). We have shown that the assumptions behind these implications, concerning working within a student's ZPD, are flawed.

The question is not whether ZPD is a useful construct for mathematics teachers and mathematics education researchers. Given the historical focus of ZPD on language acquisition, the question is when and how teachers should provide assistance within a students' *mathematical* ZPD. Some neo-Vygotskians have already recognized the dangers of applying assumptions about language acquisition to mathematical learning and development. Goos et al. (2002) warned that scaffolding techniques, when applied to mathematics teaching, sometimes interfere with problem solving. Furthermore, Bliss, Askew, and Macrae (1996) showed "a need for differentiated analyses" of scaffolding (or assisting) in students' development of mathematical objects. The former study begins to answer the question of *when* to assist; the latter study suggests, as we have, that teachers need to build models of students' operations in order to decide *how* to assist.

Our characterization of ZPD challenges the literal method of determining developmental readiness. Whereas our study affirms that teacher assistance is critical to learning and development, developmental readiness cannot be determined by assisted performance alone. ZPDs need to account for students' ways of operating,

which are determined based on *inferences* about students' performances. Teachers make inferences and build models by assuming that student actions represent a consistent and coherent way of operating and by asking, "How must this students' mathematics operate to produce these actions?" Teachers test these models by posing new tasks and checking whether student responses fit the model. Then, subsequent tasks and questions can be designed, based on this model, to assist further learning.

We have described a mathematical ZPD that is very ZPC-like. Likewise, we have described successful forms of assistance within ZPC that are very ZPD-like. Although the epistemological roots of the two constructs differ considerably, we argue that radical and social constructivist implications for teaching (in particular, regarding ZPD and ZPC) are compatible. Piaget (1950, 1970) recognized the critical importance of language and social interaction in learning, and Vygotsky (1978, 1986) recognized the importance of the internal plane of development. The latter recognition opens the door for neo-Vygotskians to model the objects and operations that might exist on the internal planes of our students, whereas the former reminds teachers to examine their own actions in supporting development.

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